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Note

A logic for rough sets

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Abstract

The collection of all subsets of a set forms a Boolean algebra under the usual set-theoretic operations, while the collection of rough sets of an approximation space is a regular double Stone algebra (Pomykala and Pomykala, 1988). The appropriate class of algebras for classical propositional logic are Boolean algebras, and it is reasonable to assume that regular double Stone algebras are a class of algebras appropriate for a logic of rough sets. Using the representation theorem for these algebras by Katriňák (1974), we present such a logic for rough sets and its algebraic semantics in the spirit of Andréka and Németi (1994).

1. Introduction

Rough set data analysis has been developed by Pawlak and his co-workers since the early 1980s as a method of dealing with coarse information. We invite the reader to consult the monographs by Pawlak [23] and Słowiński [26] for an in-depth exposition of the theory and its applications.

The building blocks of rough set analysis are *approximation spaces* $\langle U, \theta\rangle$, where U is a set, and θ is an equivalence relation on U. The intuition behind this is that objects in U can only be distinguished up to their equivalence class, and that objects within one class are indistinguishable with the information at hand. With each approximation space $\langle U, \theta\rangle$ two operators on $\mathfrak{P}(U)$ are associated: If $X \subseteq U$, then

$$\overline{X} := \bigcup\{\theta x : \theta x \cap X \neq \emptyset\}$$

is the *upper approximation of* X, *and*

$$\underline{X} := \bigcup\{\theta x : \theta x \subseteq X\}$$

is its *lower approximation*; for each $X \subseteq U$, a *rough set* is a pair $\langle \underline{X}, \overline{X}\rangle$.

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We note in passing that the upper (lower) approximation in $\langle U, \theta\rangle$ is the closure (interior) operator of the topology on U whose non-empty open sets are the union of equivalence classes. This topology is sometimes called *Pawlak topology* e.g. in [17]. The terminology seems somewhat unfortunate: It has been known for some time that on a finite set U there are natural correspondences between

- The set of all equivalence relations on U,
- The set of topologies on U in which each closed set is open,
- The set of all regular (not necessarily T_1) topologies on U,

see, for example, [15] and the references therein.

It is easily seen that the two approximation operators can be regarded as modal operators $\Diamond$ and $\Box$ induced by the frame $\langle U, \theta\rangle$. Thus, one can associate a modal **S5** logic with this concept; this is the approach taken by Orlowska [19].

Another approach is to use the algebraic structure of rough sets in analogy to the correspondence between Boolean algebras and classical propositional logic. It was shown by Pomykala and Pomykala [24] that the collection $\mathfrak{P}_\theta(U)$ of rough sets of $\langle U, \theta\rangle$ can be made into a Stone algebra $\langle \mathfrak{P}_\theta(U), +, \cdot, ^*, \langle\emptyset,\emptyset\rangle, \langle U, U\rangle\rangle$ by defining

$$\langle \underline{X}, \overline{X}\rangle + \langle \underline{Y}, \overline{Y}\rangle = \langle \underline{X} \cup \underline{Y}, \overline{X} \cup \overline{Y}\rangle,$$

$$\langle \underline{X}, \overline{X}\rangle \cdot \langle \underline{Y}, \overline{Y}\rangle = \langle \underline{X} \cap \underline{Y}, \overline{X} \cap \overline{Y}\rangle,$$

$$\langle \underline{X}, \overline{X}\rangle^* = \langle -\overline{X}, -\overline{X}\rangle,$$

where for $Z \subseteq U$, the complement of Z in U is denoted by $-Z$.

This was improved by Comer [7], who noticed that $\mathfrak{P}_\theta(U)$ is, in fact, a regular double Stone algebra when one defines the dual pseudocomplement $^+$ by

$$\langle \underline{X}, \overline{X}\rangle^+ = \langle -\underline{X}, -\underline{X}\rangle.$$

These algebras of rough sets have the special form of what we call below a *Katriňák algebra.*

We use these facts to present a propositional logic for rough sets with an algebraic semantics and exhibit some of its properties using the algebraic tools of the general approach to logic, where the semantics of formulas is determined by a general "meaning" function, and not only by a truth value assignment

$$v : \text{Formulas} \to \text{Truth values}.$$

This framework for logic is described in somewhat simplified form in [14, p. 255ff], where it is credited to a series of papers by Andréka–Németi–Sain [1, 25, 18, 4]; for a recent exposition we refer the reader to [2]. A related view has later been put forward – for a different purpose – by Epstein [13] as "set assignment semantics".

There are several closely related, resp. equivalent, constructions in algebraic logic: Using the correspondence of regular double Stone algebras to three-valued Łukasiewicz algebras (and thus to Moisil's algebras) which was first observed by Varlet [28], our results can also be considered a semantic approach to three-valued Łukasiewicz logic.

Furthermore, since these algebras are equivalent to semi–simple Nelson algebras, the rough set logic is also equivalent to "classical logic with strong negation", see [27].

The interested reader is invited to consult [22] for an exposition of the algebraic connections between rough set algebras, semi-simple Nelson algebras, three-valued Łukasiewicz algebras, and related structures, as well as [11] for an overview of the algebraic properties of rough sets.

A logic for information systems has been presented by Orlowska [20], and Comer [6] provides a class of algebras of information systems which are closely related to cylindric algebras.

2. Definitions and notation

We assume familiarity with the basic concepts of lattice theory, universal algebra, and logic. For definitions not explained here we refer the reader to [5] for lattice theory and universal algebra, and to [2] for logic. Here, we just define regular double Stone algebras which may not be widely known.

A double Stone algebra $\langle L, +, \cdot, ^*, ^+, 0, 1\rangle$ is an algebra of type $\langle 2, 2, 1, 1, 0, 0\rangle$ such that

(i) $\langle L, +, \cdot, 0, 1\rangle$ is a bounded distributive lattice,

(ii) x^* is the pseudocomplement of x, i.e.

$$y \leqslant x^* \Leftrightarrow y \cdot x = 0,$$

(iii) x^+ is the dual pseudocomplement of x, i.e.

$$y \geqslant x^+ \Leftrightarrow y + x = 1,$$

(iv) $x^* + x^{**} = 1, \quad x^+ \cdot x^{++} = 0.$

Conditions (ii) and (iii) are equivalent to the equations

$$x \cdot (x \cdot y)^* = x \cdot y^*, \qquad x + (x + y)^+ = x + y^+,$$

$$x \cdot 0^* = x, \qquad x + 1^+ = x,$$

$$0^{**} = 0, \qquad 1^{++} = 1,$$

so that the double Stone algebras form an equational class [29]. L is called *regular*, if it additionally satisfies the equation

$$x \cdot x^+ \leqslant y + y^*.$$

This is equivalent to

$$x^+ = y^+ \text{ and } x^* = y^* \text{ imply } x = y.$$

Let B be a Boolean algebra, F be a filter on B, and

$$\langle B, F\rangle := \{\langle a, b\rangle : a, b \in B, a \leqslant b, \ a + (-b) \in F\}.$$

If $B=F$, then we shall usually write $B^{[2]}$ for $\langle B,F\rangle$. We now define the following operations on $\langle B,F\rangle$:

$$\langle a,b\rangle + \langle c,d\rangle = \langle a+c, b+d\rangle,$$

$$\langle a,b\rangle \cdot \langle c,d\rangle = \langle a\cdot c, b\cdot d\rangle,$$

$$\langle a,b\rangle^{*} = \langle -b,-b\rangle,$$

$$\langle a,b\rangle^{+} = \langle -a,-a\rangle.$$

The operations on the right-hand side are the operations on B. We shall call algebras of this form *Katriňák algebras*. This is motivated by the following representation theorem.

Theorem 1 (Katriňák [16]). *Each Katriňák algebra is a regular double Stone algebra. Conversely, each regular double Stone algebra is isomorphic to a Katriňák algebra.*

If $\mathbb{S}=\langle B,F\rangle$ is a Katriňák algebra, we can identify B with $\{\langle a,a\rangle : a\in B\}$, and F with $\{\langle a,1\rangle : \langle a,1\rangle\in\mathbb{S}\}$.

We denote the category of Katriňák algebras by **K**, and the variety generated by **K** by $\mathbf{V}_{\mathbf{K}}$. Theorem 1 shows that $\mathbf{V}_{\mathbf{K}}$ is the variety of regular double Stone algebras.

The connection to the rough sets can now be described as follows.

If $\langle U,\theta\rangle$ is an approximation space, we can view the classes of θ as atoms of a complete subalgebra of the Boolean algebra $\mathfrak{P}(U)$. Conversely, any atomic complete subalgebra B of $\mathfrak{P}(U)$ gives rise to an equivalence relation θ on U via

$x\theta y :\Leftrightarrow x$ and y are contained in the same atom of B,

and this correspondence is bijective. If $\{a\}\in B$, then, for every $X\subseteq U$ we have

If $a\in\underline{X}$, then $a\in X$,

and the rough sets of the corresponding approximation space are the elements of the regular double Stone algebra $\langle B,F\rangle$, where F is the filter of B which is generated by the union of singleton elements of B

Homomorphisms of Katriňák algebras were described in [9].

Theorem 2. *Let $\mathbb{L}=\langle B,F\rangle$ and $\mathbb{M}=\langle C,G\rangle$ be Katriňák algebras, and $f:\mathbb{L}\to\mathbb{M}$ a homomorphism. Then, $f\restriction B$ is a Boolean homomorphism for which $f[F]\subseteq G$. Conversely, if $h:B\to C$ is a Boolean homomorphism with $h[F]\subseteq G$, then h can be uniquely extended to a* **K***-homomorphism $\mathbb{L}\to\mathbb{M}$.*

An *epimorphism* is a morphism f such that for all morphisms g,h

$g\circ f = h\circ f$ implies $g=h$;

see [5]. The following will be used below to describe a property of our logic.

Corollary 3. *$\mathbf{K}$ contains non-surjective epimorphisms.*

Proof. Let $\mathbb{S}=\langle B,F\rangle, B\neq F$, $\mathbb{T}=\langle B,B\rangle\in\mathbf{K}$, and $f:B\to B$ be the identity. Since $f[F]=F\subseteq B$, f can be uniquely extended to a $\mathbf{K}$-morphism $\overline{f}:\mathbb{S}\to\mathbb{L}$, and it follows from $F\neq B$ that $\overline{f}$ is not surjective. Let $\overline{g},\overline{h}:\mathbb{L}\to\mathbb{M}=\langle C,G\rangle$ be $\mathbf{K}$-morphisms with $\overline{g}\circ\overline{f}=\overline{h}\circ\overline{f}$, and g,h be the respective restrictions to B. By Theorem 2 it suffices to show that $g=h$:

$$\begin{aligned}\overline{g}\circ\overline{f}=\overline{h}\circ\overline{f} &\Rightarrow g\circ f=h\circ f,\\ &\Rightarrow g(f(x))=h(f(x)) \text{ for all } x\in B,\\ &\Rightarrow g(x)=h(x),\end{aligned}$$

which proves our claim. □

3. Rough set logic

The language of rough set logic $\mathscr{L}$ consists of a nonempty set P of propositional variables, two binary connectives $\wedge,\vee$, two unary connectives $^*,^+$ which represent two forms of negation, as well as the constant $\top$ which represents truth. Formulas are built from the propositional variables in the usual recursive way, so that the set Fml of $\mathscr{L}$-formulas with these operations becomes an absolutely free algebra of type $\langle 2,2,1,1,0\rangle$, generated by the elements of P.

A model of $\mathscr{L}$ is a pair $\langle W,v\rangle$, where W is a set, and $v:P\to\mathfrak{P}(W)\times\mathfrak{P}(W)$ a mapping – called the *valuation function* – for which for all $p\in P$,

If $v(p)=\langle A,B\rangle$, then $A\subseteq B$.

The equation $v(p)=\langle A,B\rangle$ expresses that

p holds at all states of A, and does not hold at any state outside B.

The following characterisation of *valuation* demonstrates the relationship to three-valued Łukasiewicz logic: For each $p\in P$ let $v_p:W\to\mathbf{3}=\{0,\frac{1}{2},1\}$ be a mapping. Then $v:P\to\mathfrak{P}(W)^{[2]}$ defined by

$$v(p)=\langle\{w\in W: v_p(w)=1\},\{w\in W: v_p(w)\neq 0\}\rangle$$

is a valuation.

Conversely, let v be a valuation and for each $p\in P$, let $v_p:W\to\mathbf{3}$ be defined as follows: If $v(p)=\langle A,B\rangle$, then

$$v_p(w)=\begin{cases}1 & \text{if } w\in A,\\ \frac{1}{2} & \text{if } w\in B\backslash A,\\ 0 & \text{otherwise.}\end{cases}$$

If $t: P \to \mathfrak{P}(W)^{[2]}$ is defined by

$$t(p) = \{\langle \{w \in W : v_p(w) = 1\}, \{w \in W : v_p(w) \neq 0\}\rangle\},$$

then it is easily seen that $v = t$.

Given a model $\mathfrak{W} = \langle W, v \rangle$, we define its meaning function $\mathsf{mng}: \mathsf{Fml} \to \mathfrak{P}(W) \times \mathfrak{P}(W)$ as an extension of the valuation v as follows:

$$\mathsf{mng}(\to p) = \langle W, W \rangle. \tag{1}$$

For each $p \in P$,

$$\mathsf{mng}(p) = v(p). \tag{2}$$

If $\mathsf{mng}(\varphi) = \langle A, B \rangle$ and $\mathsf{mng}(\psi) = \langle C, D \rangle$ then

$$\mathsf{mng}(\varphi \wedge \psi) = \langle A \cap C, B \cap D \rangle, \tag{3}$$

$$\mathsf{mng}(\varphi \vee \psi) = \langle A \cup C, B \cup D \rangle, \tag{4}$$

$$\mathsf{mng}(\varphi^*) = \langle -B, -B \rangle, \tag{5}$$

$$\mathsf{mng}(\varphi^+) = \langle -A, -A \rangle. \tag{6}$$

Here, $-A$ is the complement of A in $\mathfrak{P}(W)$.

Let $\operatorname{ran}(\mathsf{mng}) := \{\mathsf{mng}(\varphi) : \varphi \in \mathsf{Fml}\}$. We define operations on $\operatorname{ran}(\mathsf{mng})$ in the obvious way:

$$\mathsf{mng}(\varphi) \cdot \mathsf{mng}(\psi) = \mathsf{mng}(\varphi \wedge \psi),$$

$$\mathsf{mng}(\varphi) + \mathsf{mng}(\psi) = \mathsf{mng}(\varphi \vee \psi),$$

$$\mathsf{mng}(\varphi)^* = \mathsf{mng}(\varphi^*),$$

$$\mathsf{mng}(\varphi)^+ = \mathsf{mng}(\varphi^+).$$

Strictly speaking we should write $\mathsf{mng}_{\mathfrak{W}}$ instead of mng, since it is dependent on $\mathfrak{W}$. However, we shall omit the subscript if the choice of $\mathfrak{W}$ is clear.

Theorem 4. *With these operations,* $\operatorname{ran}(\mathsf{mng})$ *is a Katriňák algebra, and* mng *is a homomorphism.*

Proof. Set $\mathbb{S} = \operatorname{ran}(\mathsf{mng})$. It is enough to find a Boolean algebra $B \leqslant \mathfrak{P}(W)$ and a filter F of B, such that $\mathbb{S} = \langle B, F \rangle$. Let $B := \{A : \langle A, A \rangle \in \mathbb{S}\}$. Then, $W \in B$ by $\mathsf{mng}(Tp) = \langle W, W \rangle$, and $\emptyset \in B$ by $\mathsf{mng}(Tp)^* = \langle \emptyset, \emptyset \rangle$. Clearly, $\mathbb{S}$ is a 0–1 sublattice of $\mathfrak{P}(W) \times \mathfrak{P}(W)$, and thus, B is closed under $\cap$ and $\cup$. If $\mathsf{mng}(\varphi) = \langle A, A \rangle \in B$, then $\mathsf{mng}(\varphi^+) = \mathsf{mng}(\varphi)^+ = \langle -A, -A \rangle$, and thus, B is closed under complementation.

Now, let $F := \{C : \langle C, W \rangle \in \mathbb{S}\}$. If $\langle C, W \rangle, \langle D, W \rangle \in \mathbb{S}$, then $\langle C \cap D, W \rangle \in \mathbb{S}$, and thus, F is closed under $\cap$. Finally, let $C \in F$, $D \in B$, and $C \subseteq D$. Then, there are

$\varphi, \psi \in \mathsf{Fml}$ such that $\mathsf{mng}(\varphi) = \langle C, W\rangle$ and $\mathsf{mng}(\psi) = \langle D, D\rangle$. Now, $\mathsf{mng}(\varphi \vee \psi) = \langle C \cup D, W \cup D\rangle = \langle D, W\rangle$ shows that $D \in F$.

The fact that mng is a homomorphism follows immediately from the definitions. □

Observe that the homomorphism condition says that $\mathscr{L}$ is truth-functional, i.e. it satisfies Frege's compositionality principle.

The class of all models of $\mathscr{L}$ is denoted by Mod. A formula φ *holds in a model* $\mathfrak{W} = \langle W, v\rangle$, written as $\mathfrak{W} \models \varphi$, if $\mathsf{mng}(\varphi) = \langle W, W\rangle$. A set Σ of sentences *entails* a formula φ, if every model of Σ is a model of φ.

We define additional operations on Fml by

$$\varphi \to \psi := \varphi^* \vee \psi \vee (\varphi^+ \wedge \psi^{**}),$$
$$\varphi \leftrightarrow \psi := (\varphi \to \psi) \wedge (\psi \to \varphi).$$

Theorem 5. *If* $\mathfrak{W} = \langle W, v\rangle \in \mathsf{Mod}$ *and* $\varphi, \psi \in \mathsf{Fml}$, *then*

(i) $\mathfrak{W} \models \varphi \leftrightarrow \psi$ *if and only if* $\mathsf{mng}(\varphi) = \mathsf{mng}(\psi)$.

(ii) $\mathfrak{W} \models \to p \leftrightarrow \varphi$ *if and only if* $\mathfrak{W} \models \varphi$.

(iii) *If h is a homomorphism of* Fml *and* $\mathfrak{W} = \langle W, v\rangle \in \mathsf{Mod}$, *then there exists some* $\mathfrak{M} \in \mathsf{Mod}$ *such that* $\mathsf{mng}_{\mathfrak{M}} = \mathsf{mng}_{\mathfrak{W}} \circ h$.

Proof. (i) Let $\mathsf{mng}(\varphi) = \langle A, B\rangle$ and $\mathsf{mng}(\psi) = \langle C, D\rangle$. Then,

$$\begin{aligned}\mathsf{mng}(\varphi \to \psi) &= \mathsf{mng}(\varphi)^* + \mathsf{mng}(\psi) + \mathsf{mng}(\varphi)^+ \cdot \mathsf{mng}(\psi)^{**}\\ &= \langle -B, -B\rangle + \langle C, D\rangle + \langle -A, -A\rangle \cdot \langle D, D\rangle\\ &= \langle -B \cup C \cup (D \cap -A), -B \cup D\rangle.\end{aligned}$$

"⇒": If $\mathsf{mng}(\varphi \leftrightarrow \psi) = \langle W, W\rangle$, then

$$-B \cup C \cup (D \cap -A) = W$$

and, using $A \subseteq B$,

$$C \cap A = W \cap A = A.$$

Furthermore, $-B \cup D = W$ implies $B \subseteq D$, so that $\mathsf{mng}(\varphi) \leqslant \mathsf{mng}(\psi)$. The proof of the other direction is analogous.

"⇐": Suppose that $A = C$ and $B = D$; then

$$\begin{aligned}\mathsf{mng}(\varphi \to \psi) &= \langle -B \cup C \cup (D \cap -A), -B \cup D\rangle\\ &= \langle -B \cup A \cup (B \cap -A), -B \cup B\rangle\\ &= \langle W, W\rangle,\end{aligned}$$

again using $A \subseteq B$. The proof of the other direction is analogous.

(ii) This follows immediately from (i) and the definition of mng. The condition says that we have one designated truth value, namely, $\top$.

(iii) Define $m : P \to \mathfrak{P}(W)^{[2]}$ by $m = \mathsf{mng}_{\mathfrak{M}} \circ h$. Since h and mng are homomorphisms, $\langle W, m \rangle$ is the desired model. $\square$

Define the *algebraic counterpart of* $\mathscr{L}$ as

$$\mathsf{Alg}_m(\mathscr{L}) := \{\operatorname{ran}(\mathsf{mng}_{\mathfrak{W}}) : \mathfrak{W} \in \mathsf{Mod}\}. \tag{7}$$

Theorem 6. *The variety generated by* $\mathsf{Alg}_m(\mathscr{L})$ *is the variety of regular double Stone algebras.*

Proof. Let $\mathbf{V}$ be the variety generated by $\mathsf{Alg}_m(\mathscr{L})$. Each algebra of $\mathbf{V}$ is a regular double Stone algebra by Theorem 4.
Conversely, let $W = \mathbf{2}$, and $\mathbb{S} = \{\langle 0,0 \rangle, \langle 0,1 \rangle, \langle 1,1 \rangle\}$. Define $v : P \to \mathbb{S}$ by $v(p) = \langle 0,1 \rangle$ for all $p \in P$. Then, $\operatorname{ran}(\mathsf{mng}) = \mathbb{S} \cong \mathbf{3}$. It is well known that $\mathbf{3}$ generates the variety of regular double Stone algebras, which proves our claim. $\square$

It follows that $\mathscr{L}$ is a strongly nice general logic in the sense of Andréka et al. [2, 3.1 and 3.2]. Strictly speaking, we should differentiate between rough set logics with sets of propositional variables of different size κ. However, Theorem 6 shows that this is not necessary since the varieties are the same regardless of κ.

Finally, using the translations

Logic $\to$ Algebra $\to$ Logic

developed in [2], we mention some properties of $\mathscr{L}$:

Theorem 7. (i) *$\mathscr{L}$ has a finitely complete and strongly sound Hilbert style axiom system.*

(ii) *$\mathscr{L}$ has a compactness theorem.*

(iii) *$\mathscr{L}$ does not have the Beth definability property.*

Proof. (i) and (ii) follow from the fact that $\mathbf{K}$ generates a finitely axiomatizable variety and 3.11 and 3.20 of [2]. (iii) follows from Corollary 3 and 2.24 of [3]. $\square$

4. Outlook

The approach taken in the previous section can be generalized in various ways. If we keep the definition of rough set as above, but add additional structure to the object set U in such a way that our objects are binary relations on some set, we arrive at algebras of rough relations, introduced by Comer [7] and further studied by Düntsch [10]. It would be interesting to know whether the Katriňák representation of algebras of rough relations can serve as an adequate algebraic counterpart to a rough generalization of arrow logics and their relatives as described in [2]. A practical application of these concepts can be found in preference modeling, investigations into which we are currently undertaking in [12].

In another direction we add "granularity to uncertainty" in the following sense:
Let δ be a complete linear order type, and $\mathfrak{A} = \langle W, A, +, \cdot, m_i, 0, 1\rangle_{i \in \delta}$ be an algebra with the following properties:

(i) Elements of A have the form $\langle X_i \rangle_{i\in\delta}$, where $X_i \subseteq W$, and $i < j < \delta$ implies $X_i \subseteq X_j$.

(ii) $\langle A, +, \cdot, 0, 1\rangle$ is a 0–1 sublattice of ${}^{\delta}\mathfrak{P}(W)$.

(iii) For each $\alpha \in \delta$ and each $\langle X_i \rangle_{i\in\delta}$, we have $m_\alpha(\langle X_i \rangle_{i\in\delta}) = \langle Y_i \rangle_{i\in\delta}$, where $Y_i = -X_\alpha$ for all $i \in \delta$.

In analogy to rough sets we call the elements of $\mathfrak{A}$ *δ-rough sets*. These are interpreted in such a way that δ measures the relative degree of uncertainty. It may be interesting to explore these algebras and their relationship to δ-valued Łukasiewicz algebras and fuzzy sets.

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